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A mathematical model for the management of an Invalidity Old Age Survivor Pension Fund: The Exact Individual Trajectories Method

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Abstract. In the article a representation theorem for a management method of a Pension Fund is proved. This method, called *The Exact Individual Trajectories Method (EIT)* previously introduced by one of the authors, has the peculiarity of being set up on an axiomatic basis and it is an alternative to the already known management methods of actuarial present value and of stochastic trajectories.

EIT is worked out on an individual basis which takes into account, for each insured, the set of all possible future life events, called *feasible trajectories*.

The theorem we prove in this article enables us, through the representation model, to identify directly the feasible trajectories without having to consider all the potential ones, which would be numerically non-polynomial, that is beyond control.

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1 Introduction

On an axiomatic basis we formalize a financial model for the management of an *Invalidity Old Age Survivor Pension Fund IOS* (in Italy called IVS, Invalidità Vecchiaia Superstiti) with a methodology which is alternative to those based on actuarial present value and stochastic trajectories now in use.

It is an individually-based method which takes into account for each insured "the set of all possible future life events" of his position, i.e. the feasible trajectories, formally expressed in terms of *states/events*. The space of states is provided with a total ordering relative to their irreversible occurrence all through the insured person's life span. To each feasible trajectory in space, a single probability in Markovian hypothesis is given (see [1], [10] and [11]).

The Exact Individual Trajectories Method takes into account the transformation of a feasible trajectory into the contribution/benefit vector in order to assess, on an individual or collective basis, the insured person's relative equilibrium premiums.

A three-dimensional model of feasible trajectories is also introduced and through a representation theorem, a biunique correspondence of each feasible trajectory with an integer co-ordinate point of the model in question is given.

2 Preliminary statements: notation and axiomatic basis

The Exact Individual Trajectories Method, which was previously presented in its basic ideas in [2] and [3] and here is elaborated, is worked out starting from the vital cycle of the insured person's position, making use of the following notation and axiomatic basis.

The *condition* of the insured person is defined through *5 states*, each having annual validity, based on the assumption that changes (in state) take place at the beginning of the year. Let us consider the states space

$$S = \{A, P_{\mathrm{i}}, P_{\mathrm{se/o}}, P_{\mathrm{su}}, E\}$$

where:

A = active
P_{i} = invalidity pension
$P_{\mathrm{se/o}}$ = seniority and old age pension
P_{su} = survivor pension to be grouped in (according to the position of the legal predecessor):

- $P_{\mathrm{su}}^{\mathrm{ind}}$ = indirect pension (received by the death of an active)
- $P_{\mathrm{su}}^{\mathrm{i}}$ = survivor invalidity pension
- $P_{\mathrm{su}}^{\mathrm{se/o}}$ = survivor seniority and old age pension

E = elimination of the insured person's pension benefits.

In the space S let us consider the following ordering

$$A \leq P_{\mathrm{i}} \leq P_{\mathrm{se/o}} \leq P_{\mathrm{su}} \leq E.$$

For each insured we shall use the following notation:

x: age in the year $i = 0$
h: years of contribution in the year $i = 0$
X: old age pension initial year
H: age of retirement on the basis of working years
$T = \min(X - x, H - h)$: number of years needed to be entitled to old age pension benefits in the year $i = 0$
$\omega - 1$: maximum reachable age.

To simplify the matter, suppose the maximum duration in the position, also with the possible survivor pension, to coincide with the insured's maximum life span.

We define as trajectory (depending on context):

a) the vector of $\omega - x + 1$ components where the i^{th} one, is the insured person's state in the condition of insured after i years;

or

b) the function $\Pi(\cdot)$ from the ordered set of natural numbers $M : \{0, 1, 2, \ldots, \omega - x\}$ to the ordered set of states S, that's the application linking each year beginning from the present state which corresponds with $i = 0$, to one of the five states.

We therefore give the following ***axiomatic basis*** A_s, for $s = 1, \ldots, 5$, for the definition of feasible trajectories (to be intended as in b)):

$$A_1 : i, j \in M \text{ and } i \leq j \Rightarrow \Pi(i) \leq \Pi(j),$$

this axiom, therefore, provides a non-decreasing condition;

$$A_2 : i \in M \text{ and } T \leq i \Rightarrow \Pi(i) \neq A,$$

in other words, after T or more years one can't be in an active state;

$$A_3 : i \in M \text{ and } i < T \Rightarrow \Pi(i) \neq P_{\mathrm{a/v}},$$

i.e. it is not possible to be entitled to IOS benefits before T years;

$$A_4 : \exists i : \Pi(i) = P_{\mathrm{i}} \Rightarrow \forall j, j \geq i \quad \Pi(j) \neq P_{\mathrm{a/v}},$$

that is from an invalidity pension one can't step over to the condition of old age pension;

$$A_5 : \Pi(0) = A, \qquad \Pi(\omega - x) = E,$$

specifically the first component of the vector corresponds to the active state and the last to elimination state.

3 Representation theorem for feasible trajectories

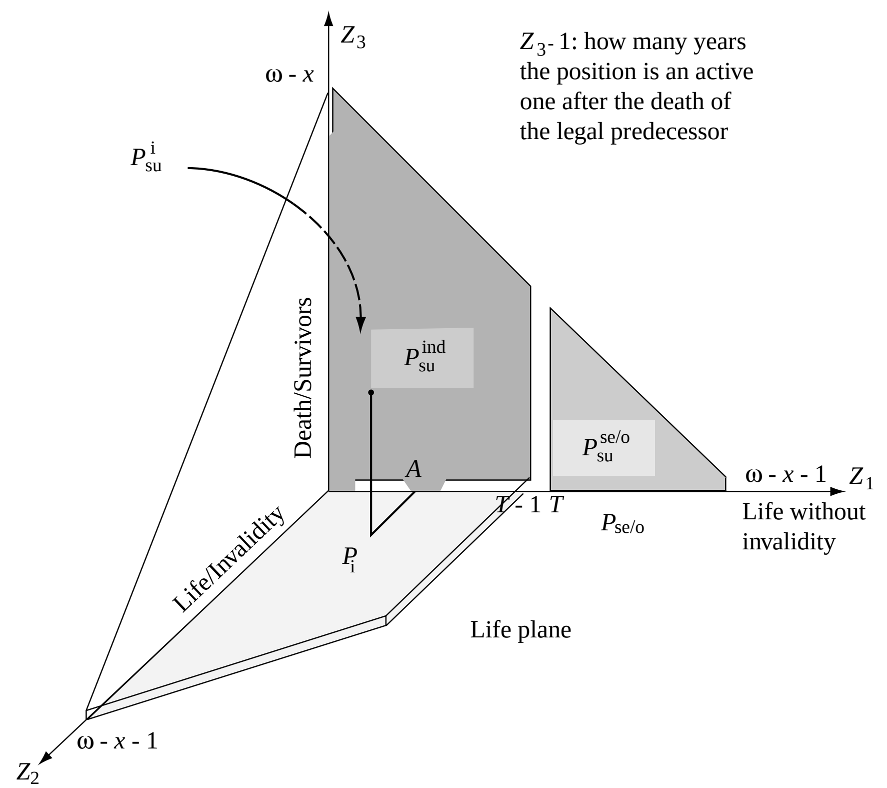
Let us consider the set $Q \subseteq R^3$ formed by points $P(z_1, z_2, z_3)$ verifying the following conditions (see also Figure 1):

$$\begin{cases} I) & z_i \text{ integer } \geq 0 \text{ for } i = 1, 2, 3 \\ \alpha) & z_1 + z_2 + z_3 \leq \omega - x \\ \beta) & z_1 \geq T \rightarrow z_2 = 0 \\ \gamma) & z_3 \geq 1. \end{cases} \tag{1}$$

Let's link each point $P(z_1, z_2, z_3) \in Q$ to a trajectory that is define an application $\Pi[P(z_1, z_2, z_3)](\cdot) : M \rightarrow S$ in the following way: using $P(z_1, z_2, z_3)$ let's divide the index set $M = [0, 1, \ldots, \omega - x]$ into the disjoint set of the index subsets M_k for $k = 1, \ldots, 5$, defined as follows (let's take $1 < T < \omega - x$):

$$\begin{cases} M_1 = \{i \in M : 0 \leq i \leq z_1 \wedge (T-1)\} \\ M_2 = \{i \in M : z_1 \wedge (T-1) < i \leq z_1\} \\ M_3 = \{i \in M : z_1 < i \leq z_1 + z_2\} \\ M_4 = \{i \in M : z_1 + z_2 < i \leq z_1 + z_2 + z_3 - 1\} \\ M_5 = \{i \in M : z_1 + z_2 + z_3 - 1 < i \leq \omega - x\}, \end{cases} \tag{2}$$

where $a \wedge b = \min\{a, b\}$.



Each feasible trajectory is in one-to-one correspondence with a point verifying I, α, β, γ representing its end component whose coordinates outline sequentially the trajectory's 'segments' (see also (1))

Figure 1: Space model of all feasible trajectories

Each index $i \in M$ belongs necessarily to one and only one of the sets M_k.

REMARK 1 We notice that the index sets M_1 and M_5 are necessarily the unique non-empty, since index $0 \in M_1$ and index $(\omega - x) \in M_5$.

REMARK 2 We observe that $M_2 \neq \emptyset$ implies $z_1 \geq T$, that for β of (1) implies $z_2 = 0$ that in its turn, equals $M_3 = \emptyset$.

As a result $M_2 \neq \emptyset \rightarrow M_3 = \emptyset$ and in an equivalent way: $M_3 \neq \emptyset \rightarrow M_2 = \emptyset$.

Consequently only one, at most, of the two index sets M_2 and M_3 linked to point $P(z_1, z_2, z_3)$ can be non-empty.

Let's therefore link to point $P(z_1, z_2, z_3) \in Q$ a trajectory, that is define an application $\Pi[P(z_1, z_2, z_3)](\cdot)$ from M into the ordered space of states S, in the following way:

$$\Pi[P(z_1, z_2, z_3)](i) = \begin{cases} A & \text{for } i \in M_1 \\ P_{\text{se/o}} & \text{for } i \in M_2 \\ P_{\text{i}} & \text{for } i \in M_3 \\ P_{\text{su}} & \text{for } i \in M_4 \\ E & \text{for } i \in M_5. \end{cases} \tag{3}$$

Let's prove the following

THEOREM *The points of the set Q are in one-to-one correspondence with the feasible trajectories, using the application defined by the relations* (2) *and* (3).

Let's first verify that the trajectory linked to each point of set Q by means of relations (2) and (3) is actually feasible.

In fact if $P(z_1, z_2, z_3) \in Q$, then the corresponding application $\Pi[P(z_1, z_2, z_3)](\cdot) : M \to S$ verifies the five axioms A_s, for $s = 1, \ldots, 5$.

TEST OF AXIOM A_1

Let us verify that, i and $j \in M$ and $i \leq j \Rightarrow \Pi[P(z_1, z_2, z_3)](i) \leq \Pi[P(z_1, z_2, z_3)](j)$. In fact, following the definition of the sets M_k, for $k = 1, \ldots, 5$, we observe that if i and j belong to M and $i \leq j$, then $i \in M_k$ and $j \in M_h$ imply that $k \leq h$.

If $i \in M_k$ and $j \in M_h$ for $k \leq h$, then considering *Remark 2*, indicating that at most one of the two index sets M_2 and M_3 may result non empty, which on the basis of the state set ordering and (3) implies that for $i \leq j$ we have

$$\Pi[P(z_1, z_2, z_3)](i) \leq \Pi[P(z_1, z_2, z_3)](j) \qquad \square$$

TEST OF AXIOM A_2

Let's verify that function $\Pi[P(z_1, z_2, z_3)](\cdot) : M \to S$ defined by relations (2) and (3) fulfils the axiom A_2, that is:

$$i \in M \text{ and } T \leq i \Rightarrow \Pi[P(z_1, z_2, z_3)](i) \neq A.$$

Hence we have:

$$z_1 \wedge (T-1) \leq T-1,$$

from which by $T \leq i$ we get

$$z_1 \wedge (T-1) \leq T-1 < T \leq i$$

and as a consequence we obtain

$$z_1 \wedge (T-1) < i.$$

The latter by (2) implies $i \notin M_1$, therefore also $\Pi[P(z_1, z_2, z_3)](i) \neq A$. □

Test of axiom A_3
We verify that, if $i \in M$ and $i < T$, then

$$\Pi[P(z_1, z_2, z_3)](i) \neq P_{\mathrm{se/o}}.$$

We observe that $\Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{se/o}}$ if and only if $i \in M_2$, and that M_2 proves non-empty if and only if $z_1 \geq T$.

In the latter case $\Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{se/o}}$ if and only if the condition

$$T - 1 < i \leq z_1$$

holds (notice that $z_1 \wedge (T - 1) = T - 1$).

When $i \in M$ the above condition is equivalent to

$$T \leq i \leq z_1.$$

As a result we have

$$\Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{se/o}} \Leftrightarrow T \leq i \leq z_1,$$

consequently, if $i < T$ then $\Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{se/o}}$ cannot take place.

In other words, the insured person can't be entitled to OAP benefits before T years have elapsed.

Test of axiom A_4
We verify that

$$\exists i \in M : \Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{i}} \Rightarrow \forall j, j \geq i, \quad \Pi[P(z_1, z_2, z_3)](j) \neq P_{\mathrm{se/o}}.$$

To this it suffices to observe that, given an index $i \in M$ with $\Pi[P(z_1, z_2, z_3)](i) = P_{\mathrm{i}}$, the index set M_3 is proved non-empty and so, on the basis of Remark 2, the index set M_2 is empty. □

Test of axiom A_5
We have to check that

$$\Pi[P(z_1, z_2, z_3)](0) = A \text{ and } \Pi[P(z_1, z_2, z_3)](\omega - x) = E.$$

It is enough to see that for any choice of point $P(z_1, z_2, z_3) \in Q$ on the basis of the definitions (2) of the index sets M_1 and M_5, we have

$$0 \in M_1 \text{ and } (\omega - x) \in M_5.$$

Hence, on the basis of (3) we have

$$\Pi[P(z_1, z_2, z_3)](0) = A \text{ and } \Pi[P(z_1, z_2, z_3)](\omega - x) = E. \qquad \square$$

We have therefore proved that given a point $P \in Q$, the trajectory linked to it by means of definitions (2) *and* (3) *is **feasible**, in other terms, it verifies the axioms A_s, for $s = 1, \ldots, 5$.*

Let us now prove that the relations (2) and (3) define an injective and surjective (i.e. biunique) applications between sets Q and T_A (the sets of feasible trajectories).

4 Application injectivity test

Let us take two distinct points $P(z_1, z_2, z_3)$ and $P'(z'_1, z'_2, z'_3)$ belonging to Q.

We must prove that the two feasible trajectories linked with them by the defining relations (2) and (3), are different.

With respect to $P(z_1, z_2, z_3)$ we observe that

$$\begin{cases} [0, z_1] = M_1 \cup M_2 = (\Pi[P(z_1, z_2, z_3)])^{-1}(A, P_{\mathrm{se/o}}) \\ (z_1, z_1 + z_2] = M_3 = (\Pi[P(z_1, z_2, z_3)])^{-1}(P_{\mathrm{i}}) \\ (z_1 + z_2, z_1 + z_2 + z_3 - 1] = M_4 = (\Pi[P(z_1, z_2, z_3)])^{-1}(P_{\mathrm{su}}) \\ (z_1 + z_2 + z_3 - 1, \omega - x] = M_5 = (\Pi[P(z_1, z_2, z_3)])^{-1}(E). \end{cases} \tag{4}$$

We indicated as first member of relations (4) within brackets a few natural number intervals that may or may not include the corresponding last term, depending on whether the bracket is square or round. In a parallel way as to point $P'(z'_1, z'_2, z'_3)$ we have

$$\begin{cases} [0, z'_1] = M'_1 \cup M'_2 = (\Pi[P'(z'_1, z'_2, z'_3)])^{-1}(A, P_{\mathrm{se/o}}) \\ (z'_1, z'_1 + z'_2] = M'_3 = (\Pi[P'(z'_1, z'_2, z'_3)])^{-1}(P_{\mathrm{i}}) \\ (z'_1 + z'_2, z'_1 + z'_2 + z'_3 - 1] = M'_4 = (\Pi[P'(z'_1, z'_2, z'_3)])^{-1}(P_{\mathrm{su}}) \\ (z'_1 + z'_2 + z'_3 - 1, \omega - x] = M'_5 = (\Pi[P'(z'_1, z'_2, z'_3)])^{-1}(E). \end{cases} \tag{4'}$$

We have indicated with an apex the sets defined by relations (2) as to point $P'(z'_1, z'_2, z'_3)$.

It follows that, if the two points $P(z_1, z_2, z_3)$ and $P'(z'_1, z'_2, z'_3)$ differ on account of the first component, $z_1 \neq z'_1$, then by (4) and (4′) the two feasible trajectories differ from each other in the components relative to the set of the two states A and $P_{\mathrm{se/o}}$ and they are, therefore, different.

If instead, the two points differ in the second component, that is $z_2 \neq z'_2$, then the two feasible trajectories differ in the components corresponding to the state P_{i}: the 'length span' of the two sets M_3 and M'_3 is different, therefore $M_3 \neq M'_3$ (including also the case when, for one of the two trajectories, the set in question is empty, which occurs if the second component of the corresponding point is 0).

Finally, if the two points differ in the third component, that is $z_3 \neq z'_3$, then the two feasible trajectories differ in the components corresponding to P_s: the 'length span' of the two sets M_4 and M'_4 is different, therefore $M_4 \neq M'_4$ (also when, due to one of the two trajectories the set in question is empty, which occurs if the corresponding point has the third component (≥ 1 for γ of (1)) exactly equal to 1).

5 Application surjectivity test

Now we prove that the application $\Pi[P] : Q \to T_f$ (set of feasible trajectories) defined by relations (2) and (3) is *surjective*.

Let us consider a feasible trajectory, that is an application $\Pi(\cdot) : M \to S$ verifying axioms A_s, for $s = 1, \ldots, 5$ and prove that it issues from a point $P(z_1, z_2, z_3) \in Q$ by applying the relations (2) and (3).

We link the index sets corresponding to each state in view of application $\Pi(\cdot)$. We then have:

$$\begin{cases} N_1 = \{i \in M : \Pi(i) = A\} \\ N_2 = \{i \in M : \Pi(i) = P_{\mathrm{se/o}}\} \\ N_3 = \{i \in M : \Pi(i) = P_{\mathrm{i}}\} \\ N_4 = \{i \in M : \Pi(i) = P_{\mathrm{su}}\} \\ N_5 = \{i \in M : \Pi(i) = E\}. \end{cases} \tag{5}$$

We proceed by furtherly assessing the following values z_1, z_2, z_3 connected with the feasible trajectory $\Pi(\cdot)$:

$$\begin{aligned} z_1 &= \max_{i \in N_1 \cup N_2} i \\ z_2 &= \begin{cases} \max\limits_{i \in N_3} i - z_1 & \text{if} \quad N_3 \neq \emptyset \\ 0 & \text{if} \quad N_3 = \emptyset \end{cases} \\ z_3 &= i_t - (z_1 + z_2), \end{aligned} \tag{6}$$

where i_t is the index corresponding to the *last component* of the feasible trajectory, that is the first moment where state E occurs.

We want to prove that:

a) the following sets of equalities hold:

$$\begin{cases} N_1 = \{i \in M : 0 \leq i \leq z_1 \wedge (T-1)\} \\ N_2 = \{i \in M : z_1 \wedge (T-1) < i \leq z_1\} \\ N_3 = \{i \in M : z_1 < i \leq z_1 + z_2\} \\ N_4 = \{i \in M : z_1 + z_2 < i \leq z_1 + z_2 + z_3 - 1\} \\ N_5 = \{i \in M : z_1 + z_2 + z_3 - 1 < i \leq \omega - x\} \end{cases} \tag{7}$$

b) point $P(z_1, z_2, z_3)$, the coordinates of which are defined by the relations (5) and (6), belongs to the set Q.

If using (5) and (6), we apply the application defined by the relation (2) and (3) to the point $P(z_1, z_2, z_3)$ linked with the feasible trajectory $\Pi(\cdot)$, we obtain again the feasible trajectory $\Pi[P(z_1, z_2, z_3)(\cdot)]$.

We have thus proved that the feasible trajectory $\Pi(\cdot)$ is obtained by means of the application qualified by the relations (2) and (3), from $P(z_1, z_2, z_3) \in Q$, whose coordinates are defined starting from $\Pi(\cdot)$, using relations (5) and (6).

If we prove that a) and b) are true, we have also proved that the application $\Pi(P) : Q \to T_f$ is surjective and, besides, that the application defined by means of (5) and (6) from T_f to Q is the inverse of the application $\Pi(P) : Q \to T_f$ defined by the relations (2) and (3).

Let us prove that a) and b) are true.

Proof of a) Let us consider a feasible trajectory, that is an application $\Pi(\cdot) : M \to S$ verifying the axioms A_s for $s = 1, \dots, 5$. We start with

Observation 1 Generally, the sets N_K, $k = 1, \dots, 5$, if not empty (under axiom A_5 only the sets N_1 and N_5 are certainly such), *are intervals of natural numbers*, which may contain one single point. As a matter of fact, due to axiom A_1, we have:

$$\text{if } i, j \in M \text{ and } i \leq j \text{ and } \Pi(i) = \Pi(j)$$
$$\text{then } \Pi(i) = \Pi(k) = \Pi(j) \text{ for } k \in M \quad i \leq k \leq j.$$

As a result, for each N_K, $k = 1, \dots, 5$, (non-empty) and for $i \in M$, $i \in N_k$ if and only if

$$\min_{i \in N_k} i \leq i \leq \max_{i \in N_k} i \tag{8}$$

Observation 2 Owing to the options of axioms A_1 and A_4, $N_3 \neq \emptyset \Rightarrow N_2 = \emptyset$, or equivalently $N_2 \neq \emptyset \Rightarrow N_3 = \emptyset$, that is *at most one of the two sets N_2 and N_3 may result non-empty.*

Given the feasible trajectory $\Pi(\cdot) : M \to S$, we verify that equalities (7) are applicable, and we specifically start by proving that the set $N_1 = \{i \in M : \Pi(i) = A\}$ equals the set $\{i \in M : 0 \leq i \leq z_1 \wedge (T-1)\}$, with z_1 defined in (6).

Set N_1 is non-empty because $0 \in N_1$ with $0 = \min_{i \in N_1} i$ and therefore by *Observation 1*, for $i \in M$, we have $i \in N_1 \Leftrightarrow 0 \leq i \leq \max_{i \in N_1} i$.

Now we verify that

$$\max_{i \in N_1} i = z_1 \wedge (T-1). \tag{9}$$

Let's consider the two possible cases:

(i) $z_1 \leq (T-1)$.

Then due to Axiom A_3, it follows that $N_2 = \emptyset$, therefore:

$$z_1 \wedge (T-1) = z_1 = \max_{i \in N_1 \cup N_2} i = \max_{i \in N_1} i,$$

i.e. (9) is true.

(ii) $z_1 \geq T$.

Then, by axiom A_2 we have $N_2 \neq \emptyset$. Hence, according to *Observation 2*, we get $N_3 \neq \emptyset$, i.e. the state P_{i} "included" in the state ordering between A and $P_{\mathrm{se/o}}$, *is not taken by the trajectory* $\Pi(\cdot)$.

Since $N_2 \neq \emptyset$ and A_1 holds, we obtain

$$\max_{i \in N_1} i < \max_{i \in N_1 \cup N_2} i = \max_{i \in N_2} i = z_1.$$

Then by axiom A_2

$$\max_{i \in N_1} i \leq T - 1.$$

Therefore:

$$\max_{i \in N_1} i \leq T - 1 < z_1. \tag{10}$$

Now if we had

$$\max_{i \in N_1} i < T - 1$$

then an index $\bar{i}$ would exist, with $\bar{i} \leq T - 1$, such that $\Pi(\bar{i}) = P_{\mathrm{se/o}}$ (in the interval $[0, z_1]$ only the states A and $P_{\mathrm{se/o}}$ are taken), in contradiction with axiom A_3.

Therefore $\max\limits_{i \in N_1} i = T - 1$ and by (10), it follows that

$$\max_{i \in N_1} i = z_1 \wedge (T - 1),$$

i.e. (9) holds.

Therefore, if $z_1 \geq T$, the following inequalities are true

$$0 = \min_{i \in N_1} i < \max_{i \in N_1} i = z_1 \wedge (T - 1) < \min_{i \in N_2} i \leq \max_{i \in N_1 \cup N_2} i = \max_{i \in N_2} i = z_1.$$

As a result, the interval of natural numbers $[0, z_1]$ *contains* the two intervals which coincide with N_1 and N_2 (*Observation 1*) defined in terms of the states A and $P_{\mathrm{se/o}}$, respectively, i.e.

$$N_1 = \left\{ i \in M : 0 \leq i \leq \max_{i \in N_1} i = z_1 \wedge (T - 1) \right\},$$
$$N_2 = \left\{ i \in M : \min_{i \in N_2} i \leq i \leq \max_{i \in N_1 \cup N_2} i = \max_{i \in N_2} i = z_1 \right\}.$$

Considering that the state P_{i}, located between A and $P_{\mathrm{se/o}}$, is not taken by the trajectory and by axiom A_1, only the states A and $P_{\mathrm{se/o}}$ can be taken in the interval $[0, z_1]$, consequently:

$$\min_{i \in N_2} i = \left(\max_{i \in N_1} i \right) + 1.$$

Therefore N_2, that is the interval defined by state $P_{\mathrm{se/o}}$ can be formulated as

$$N_2 = \left\{ i \in M : z_1 \wedge (T - 1) = \max_{i \in N_1} i < i \leq z_1 \right\}, \tag{11}$$

and the interval $[0, z_1]$ is split into the two disjoint intervals

$$N_1 = \{i \in M : 0 \leq i \leq z_1 \wedge (T-1)\} \text{ relative to state } A,$$
$$N_2 = \{i \in M : z_1 \wedge (T-1) < i \leq z_1\} \text{ relative to state } P_{\mathrm{se/o}}.$$

As to the second equality (7) we observe that $N_2 \neq \emptyset$ if and only if $z_1 \geq T$. Therefore, following the reasoning of (ii), the set N_2, being an interval (see *Observation 1*) can be written in the form (11), i.e., the latter of the two considered equalities of (7) is true.

As for the set N_3, if it's not empty, then it is an interval, i.e. for $i \in M$ we have

$$i \in N_3 \Leftrightarrow \min_{i \in N_3} i \leq i \leq \max_{i \in N_3} i. \tag{12}$$

For $\min\limits_{i \in N_3} i$, by *Observation 2* we can see that the following sequence of implications and equivalences is true:

$$N_3 \neq \emptyset \Rightarrow N_2 = \emptyset \Leftrightarrow z_1 \leq T-1 \Leftrightarrow z_1 = z_1 \wedge (T-1) \tag{13}$$

and consequently

$$N_3 \neq \emptyset \Rightarrow z_1 = z_1 \wedge (T-1) = \max_{i \in N_1} i.$$

By axiom A_1 and the state ordering, the interval N_3, defined by the state P_{i}, follows immediately the interval N_1 defined by the state A and hence by (13), we get:

$$\min_{i \in N_3} i = \left(\max_{i \in N_1} i\right) + 1 = z_1 \wedge (T-1) + 1 = z_1 + 1.$$

By $N_3 \neq \emptyset$, from relation (6) we also have

$$\max_{i \in N_3} i = z_1 + z_2. \tag{14}$$

Hence for all $i \in M$, we obtain

$$i \in N_3 \Leftrightarrow z_1 < i \leq z_1 + z_2,$$

that is, the third equality in (7) holds.

Concerning set N_4, if non-empty, it must be an interval (as already noticed), i.e. for all $i \in M$ we have:

$$i \in N_4 \Leftrightarrow \min_{i \in N_4} i \leq i \leq \max_{i \in N_4} i.$$

By axiom A_1 and the states ordering, interval N_4 follows one of the two intervals N_2 or N_3 (if one of the two is non-empty) otherwise N_4 is next to interval N_1.

We observe that $z_1 + z_2$ in any case represents the right end point of the interval in question, either in the case it is equal to N_2 or N_3 (if both N_2 and N_3 are empty), or when it is equal to N_1.

In fact, if interval N_2 is non-empty, N_3 must be empty, and we have $z_2 = 0$, therefore $z_1 + z_2 = z_1$ which is the right end point of the interval N_2 (see the second relation of (7)).

If instead, N_3 is non-empty (implying N_2 is empty), it follows (see (14)) that $z_1 + z_2$ is the right end point of N_3. In case both N_2 and N_3 are empty, we have $z_1 = \max_{i \in N_1} i$ and $z_2 = 0$ (see (6)).

Therefore $z_1 + z_2 = \max_{i \in N_1} i$ that is $z_1 + z_2$ is the right end of interval N_1, implying $\min_{i \in N_4} i = z_1 + z_2 + 1$.

From axiom A_1, due to the ordering of the states set S and the third relation in (6), it follows:

$$\max_{i \in N_4} i = i_t - 1 = z_1 + z_2 + z_3 - 1.$$

Therefore, for $i \in M$ we have

$$i \in N_4 \Leftrightarrow z_1 + z_2 < i \leq z_1 + z_2 + z_3 - 1,$$

i.e. the fourth of the equalities (7) holds.

As to the fifth and last of the equalities (7), we observe the following: since i_t is the first component of trajectory $\Pi(\cdot)$ where the state E occurs, from axiom A_1, and taking into account the ordering set S, the state in question is present in all subsequent components of the trajectory. N_5 (which is non-empty by $(\omega - x) \in N_5$, resulting from A_5) is therefore the interval of all $i \in M$, such that:

$$z_1 + z_2 + z_3 - 1 < z_1 + z_2 + z_3 = i_t \leq i \leq \omega - x. \qquad \square$$

Test of b) We verify that point $P(z_1, z_2, z_3)$, defined by the relations (5) and (6) belongs to set Q.

Let us check I) of (1).

Indeed, for $i = 1, 2, 3$, z_1 is a non-negative integer, that's I) of (1) holds.

Obviously $z_1 \geq 0$. For z_2 we have that either it is 0, if the set N_3 proves empty, or in the case the latter is non-empty, it follows that $z_2 = \max_{i \in N_3} i - z_1$.

If the set N_3 is non-empty, then the set N_2 is empty by axioms A_1 and A_5, therefore $z_1 = \max_{i \in N_1} i$. As a result we have $z_2 = \max_{i \in N_3} i - \max_{i \in N_1} i$ which, again by axiom A_1, considering the ordering in S, implies $z_2 > 0$.

As to z_3, we observe that

$$z_3 = i_t - z_1, \text{ if } N_3 = \emptyset$$

and

$$z_3 = i_t - \max_{i \in N_3} i, \text{ if } N_3 \neq \emptyset.$$

From axiom A_1, considering that i_t is the first component of the trajectory in which state E, *the last one* of all 5 states is present, it follows $z_3 \geq 1$, i. e. also γ) of (1) holds.

Now we verify α) of (1).

Since $i_t \leq \omega - x$, owing to the third relation of (6), we have $z_1 + z_2 + z_3 = i_t \leq \omega - x$, i.e. α) of (1) fulfils.

Finally let's verify β) of (1).

If $z_1 \geq T$, then by axiom A_2 and the state ordering we have $z_1 = \max_{i \in N_2} i$ implying $N_2 \neq \emptyset$, and therefore (see *Observation 2*), $N_3 = \emptyset$ and as a consequence $z_2 = 0$.

As a result $z_1 \geq T \Rightarrow z_2 = 0$, i.e. β) of (1) is also valid. □

6 Conclusion

The axiomatic basis which, as already mentioned, is the peculiarity of the formalization by EIT method, represents a specific feature which has relevance not only for the formal transparency of this method, but because it is also an instrument of control as well as coherency of the method itself, which has proved essential in the applications.

We have in fact verified in an operative way the real importance of the representation theorem in some actuarial studies of the forecasting type.

The representation model enables us to identify immediately the feasible trajectories without having to consider all the potential ones, numerically beyond control, by eliminating the non-feasible ones.

With the mathematical model considered in this paper, we have worked out an 80 years' estimate technique balance sheet for some Italian Social Security Insurance Fund Organisations for each Professional Category (called in Italy Casse di Previdenza dei Liberi Professionisti).

These actuarial projections have allowed in some cases to estimate and carry out a Reform of the Social Security System of the above mentioned organisations of professionals.

By the classical methods either of the collective type like *subdivided in groups collectivities* (see [14] and [15]) and *multistate models* (see [1], [10], [11] and [17]) or of the individual type like *actuarial present value* (see [5], [9], [12], [16] and [17]) and *stochastic trajectories* (see [4], [6], [7], [8], [13], [16], [17] and [18]), an 80 years' projection would be extremely onerous because as to stochastic trajectories, serious difficulties may arise due both to the lack of analytical solutions relative to the structure and to the necessity to assess sensitivity via evidence.

Consequently, by the Exact Individual Trajectories Method we succeed in working out non - stochastically generated trajectories, in making a control by the axiomatic basis, thus restricting the probabilistic asset to the evaluation of the single trajectory probabilities and obtaining the problem's structure with an extremely reduced computation complexity, which permits us to single out all the possible life events conciliating methodological rigour, formal transparency and didactic simplicity.

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